A CHEMICAL BALANCE WEIGHING DESIGN

By

MEENA N. SWAMY

University of Madras, Madras-5

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SUMMARY

Four chemical balance weighing designs D_2, Γ_3, D_4 and D_5 each with 2v weighing operations have been suggested by Dey (1972) as alternative to the 'repeated' design D_1. The present paper proposes a new design D_0 with only $(2v-1)$ weighing operations, which is shown to be superior to some of the existing designs in certain cases.

1. INTRODUCTION

The results of n weighing operations to determine the individual weights of v light objects on a chemical balance with zero bias fit into the linear model $y=X\beta+e$ where $X=(x_{ij})$, $i=1, 2, ..., n$, $j=1, 2, ..., v$, is an $n\times v$ matrix of elements $x_{ij}=+1, -1$ or 0 according as in the ith weighing operation, the j-th object is placed respectively, on the left pan, right pan or none; y is the $n\times 1$ observed vector of the recorded weighings; β is the $v\times 1$ vector of the unknown weights of the objects and e is an $n\times 1$ random vector of errors with $E(e)=0$, $E(ee')=\sigma^2 I_n$ where E stands for expectation, e' denotes the transpose of e and I_n is the $n\times n$ identity matrix. X is called the weighing design.

If $X'X$ is non-singular, the least squares estimates of the weights are given by $\hat{\beta}=(X'X)^{-1}\ X'y$ with covariance matrix Cov $(\hat{\beta})=(X'\ X)^{-1}\ \sigma^2$. Henceforth we take $C=(X'\ X)^{-1}=(c_{ij})$.

The purpose of this paper is to suggest a chemical balance design which is shown to be better than some of the existing designs.

2. CRITERIA FOR THE COMPARISON OF TWO WEIGHING DESIGNS

Let v denote the number of objects whose weights are to be estimated. The following three criteria are chosen for comparing two designs.

Criterion (*i*). Of two weighing designs X_1 and X_2, X_1 is superior to X_2 if the variance of each of the estimates is smaller in the case of design X_1 than in the case of design X_2.

Criterion (*ii*). X_1 is superior to X_2 if $tr(C_1) < tr(C_2)$ where $C_1 = (X_i^1\ X_i)^{-1}$, $i=1, 2$ (A-optimality).

Criterion (*iii*). X_1 is superior to X_2 if $\det(X_1{}'X_1) > \det(X_2{}'X_2)$ (D-optimality).

3. The 'Repeated' Design and its Alternatives

3.1. *The 'repeated' design.* One of the methods of constructing a weighing design is based on the incidence matrix of a balanced incomplete block (BIB) design. The incidence matrix of a BIB design with parameters v, b, r, k, λ is the $b \times v$ matrix N^* defined by $N^* = (n_{ij})$ where $n_{ij} = 1$ or 0 according as the j-th treatment occurs or not in the ith block. By replacing 0 by -1 in N^* we can obtain a matrix N_1 which is a chemical balance weighing design to weigh v objects in b weighings. If the BIB design is symmetrical, *i.e*, if $b = v$, no degrees of freedom are left for the estimation of the error variance, To overcome this difficulty, one alternative is to repeat the design N_1 which results in

$$D_1 = \begin{bmatrix} N_1 \\ \cdots \\ N_1 \end{bmatrix}$$

We call D_1 the 'repeated' design.

3.2. *Alternatives to the 'repeated' design suggested by Dey* (*1972*). The alternatives to the 'repeated' design D_1 as given in Dey (1972) are the following :

$$D_2 = \begin{bmatrix} N_1 \\ \cdots \\ N_2 \end{bmatrix}; \quad D_3 = \begin{bmatrix} N_1 \\ \cdots \\ J_{v,v} \end{bmatrix} \text{ or } \begin{bmatrix} N_1 \\ \cdots \\ -J_{v,v} \end{bmatrix};$$

$$D_4 = \begin{bmatrix} N_1 \\ \cdots \\ N_3 \end{bmatrix} \text{ or } \begin{bmatrix} N_1 \\ \cdots \\ -N_3 \end{bmatrix}; \quad D_5 = \begin{bmatrix} N_1 \\ \cdots \\ I_v \end{bmatrix} \text{ or } \begin{bmatrix} N_1 \\ \cdots \\ -I_v \end{bmatrix}$$

where, N_2 is the incidence matrix of the complementary BIB design, $J_{v,v}$ is a $v \times v$ matrix with each element unity and N_3 is a square matrix of order v derived from I_v by replacing 0 by -1.

3.3. *The new design* D_0. Without any loss of generality let us assume that the $r(=k)$ elements of the first block B_1 of the chosen

BIB design represent the first r of the v objects to be weighed. If the block B_1 is deleted from the BIB design. we shall be left with $(v-1)$ blocks in each of which there will be exactly λ elements from among the first r objects of the BIB design. Form the $(v-1)\times v$ matrix N_0 from the incidence matrix N^* of the chosen BIB design by performing the following operations : (*i*) delete the first row of N^*, (*ii*) replace 0 by -1 and (*iii*) in each row of the resulting matrix, replace by zeros the λ unities corresponding to the λ elements the row has from among the first r objects. The design D_0 is given by

$$D_0=\begin{bmatrix} N_1 \\ \cdots \\ N_0 \end{bmatrix}.$$

It can be easily verified that

$$D_0'\,D_0=\begin{bmatrix} A & B \\ B' & D \end{bmatrix}$$

where
$$A=5\,(r-\lambda)\,I_r+(2v-6r+5\lambda)\,J_{r,\,r}$$
$$B=(2v-7r+6\lambda)\,J_{r,\,(v-r)}$$
and
$$D=8(r-\lambda)\,I_{v-r}+\{2v-1-8\,(r-\lambda)\}\,J_{(v-r)\,.\,(v-r)}$$

3.4. *The S_1 and S_2 series and the non-singularity of D_i. $i=0, 1, 2, 3, 4, 5$.* When $v\neq 2r$, N_1 is non-singular and consequently, designs D_i. $i=0, 1, 2, 3, 4, 5$, which are obtained by augmenting N_1 with additional rows, will also be non-singular.

Dey (1972) has compared the designs D_i, $i=1, 2, 3, 4, 5$ with respect to the following two series of BIB designs :

$S_1 : v=b=4t-1, \quad r=k=2t-2, \quad \lambda=t-1 \quad t\geqslant 1$

$S_2 : v=b=s^2+s+1, \quad r=k=s+1, \quad \lambda=1$ s being a prime power.

For the purpose of comparing D_0 with D_i, $i=1, 2, 3, 4, 5$, we shall restrict our studies to the above two series of BIB designs.

4. Comparisons Between the Design

Criterion (*i*). Consider the S_1 series. The expressions for the variance factors are as follows :

TABLE 1

Design	*Variance factors for the first r objects*	*Variance factors for the last (v−r) objects*
D_0	$(2t^2-t+4)/10t\,(t^2+1)$	$(2t^3+3t^2+6t+1)/16t^2\,(t^2+1)$
D_1 to D_5	*See Dey (1972)*	

It can be easily seen that D_0 is superior to all the designs D_i, $i=1, 2, 3, 4, 5$ when $t \geqslant 4$.

Next consider the S_2 series. The variance factors are seen to be as follows :

TABLE 2

Design	*Variance factors for the first r objects*	*Variance factors for the last (v−r) objects*
D_0	$\dfrac{9s^3-14s^2-27s+48}{5\,(9s^4-15s^3-11s^2+16s+8)}$	$\dfrac{9s^4-15s^3-20s^2+47s+3}{8s\,(9s^4-15s^3-11s^2+16s+8)}$
D_1 to D_5	See Dey (1972)	

Comparing the variance factors we see that D_0 is superior to all D_i's except D_1 for $s=4$, 5. For $s \geqslant 7$, D_0 is inferior to D_1, and also to D_3 for the estimation of the weights of the first r objects, but is superior to the other designs.

Criterion (*ii*). Consider the S_1 series. The values of $v^{-1}\,tr(C)$ for the various designs are as follows :

TABLE 3

Design	$v^{-1}\,tr(C)$
D_0	$(26t^3-t^2+66t-11)/40t\,(t^2+1)\,(4t-1)$
D_1	$1/4t$
D_2	$(4t^2-t+2)/5t\,(1+4t^2)$
D_3	$(4t^2-3t+1)/2t\,(8t^2-4t+1)$
D_4	$(4t^2-7t+4)/2\,(t+1)\,(8t^2-12t+5)$
D_5	$3/2\,(4t+1)$

Comparison of these values shows that D_0 is superior to all the other designs for $t \geqslant 4$.

For the S_2 series the values of $v^{-1}\, tr(C)$ are as follows :

TABLE 4

Design	$v^{-1}\, tr\,(C)$
D_0	$(45s^5-3s^4-140s^3-93s^2+183s+384)/40\,(s^2+s+1)\,(9s^4-15s^3-11s^2+16s+8)$
D_1	$(s^3-2s^2-2s+5)/8\,(s^2-s-1)^2$
D_2	$(2s^3-2s^2-3s+6)/5\,(2s^4-2s^3-s^2+2s+1)$
D_3	$(s^3+3)/4\,(s^4+s^2+2s+1)$
D_4	$(s^4-2s^2+s+2)/4\,(s+1)\,(s^4-s^2+1)$
D_5	$(s^4-2s^3-2s^2+5s+1)/(4s+1)\,(s^4-2s^6-s^2+2s+2)$

In this case it is seen that D_0 is superior to all the other designs except D_1 when $s \geqslant 3$.

Criterion (*iii*). Consider the S_1 series. The values of det $\left(D_i' D_i\right)$ are as follows :

TABLE 5

Design	$det\,(D_i'\,D_i)$
D_0	$5^{2t-2}\,2^{6t-1}\,t^{4t-2}\,(t^2+1)$
D_1	$2^{12t-5}\,t^{4t-2}$
D_2	$5^{4t-2}\,t^{4t-2}\,(4t^2+1)$
D_3	$2^{8t-3}\,t^{4t-2}\,(8t^2-4t+1)$
D_4	$2^{8t-3}\,(t+1)^{4t-2}\,(8t^2-12t+5)$
D_5	$2\,(4t+1)^{4t-2}$

Comparison of these values indicates that D_0 is superior to all the other designs except D_1 when $t \geqslant 4$.

For the S_2 series, the values of det $\left(D_i' D_i\right)$ are as follows :

TABLE 6

Design	$det\,(D_i'\,D_i)$
D_0	$5^s\,8^{s^2-1}\,S^{s^2+s}\,(9s^4-15s^3-11s^2+16s+8)$
D_1	$2^{3s^2+8s+1}.S^{s^2-s}\,(s^4-2s^3-s^2+2_3+1)$
D_2	$5^{s^2+s}.S^{s^2+s}\,(2s^4-2s^3-s^2+2s+1)$
D_3	$2^{2s^2+2s+1}.S^{s2+s}(S+s^2+2s+1)$
D_4	$2^{2s^2+2s^2+1}\,(s+1)s^{2+s}\,(s^4-s^2+1)$
D_5	$(4s+1)^{s^2+s}(s^4-2s^3-s^2+2s+2)$

Comparison of these values indicates that D_0 is superior to all the other designs except D_1 when $s \geqslant 3$

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Dey, A. (1972) : Some remarks on chemical balance weighing designs. *J. Indian Soc. Agr. Stat.*, **24**, pp. 119-126.